The Parallelogram Law

Objective: To take students through the process of discovery, making a conjecture, further exploration, and finally proof.

I. Introduction: Use **one** of the following…

- Geometer's Sketchpad demonstration
- Geogebra demonstration
- The introductory handout from this lesson

Using one of the introductory activities, allow students to explore and make a conjecture about the relationship between the sum of the squares of the sides of a parallelogram and the sum of the squares of the diagonals.

Conjecture: The sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.

Ask the question: Can we prove this is always true?

II. Activity: Have students look at one more example. Follow the instructions on the exploration handouts, "Demonstrating the Parallelogram Law."

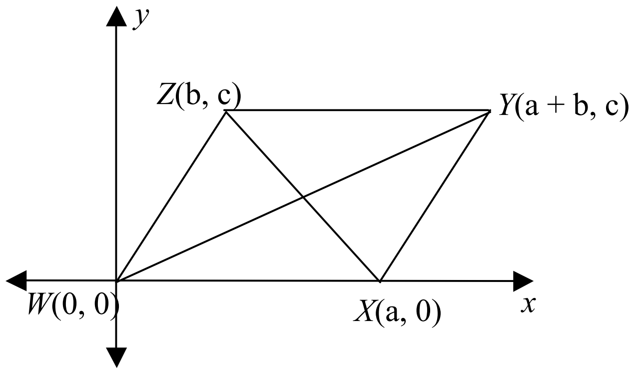
- Give each student a copy of the student handouts, scissors, a glue stick, and two different colored highlighters. Have students follow the instructions. When they get toward the end, they will need to cut very small pieces to fit in the uncovered space. Most likely there will be a very small amount of space left uncovered, or a small amount will extend outside the figure.
- After the activity, discuss the results. Did the squares along the two diagonals fit into the squares along all four sides?

  Since it is unlikely that it will fit exactly, students might question if the relationship is always true. At this point, talk about how we will need to find a convincing proof.

III. Go through one or more of the proofs below:

A. Coordinate proof:

Given: Parallelogram *WXYZ*   Prove: $WX^2 + XY^2 + YZ^2 + ZW^2 = WY^2 + XZ^2$



Side lengths:

$$WX = \sqrt{a^2 + 0^2}$$
$$= \sqrt{a^2}$$
$$= a$$

$$XY = \sqrt{(a+b-a)^2 + (c-o)^2}$$
$$= \sqrt{b^2 + c^2}$$

$$YZ = \sqrt{(a+b-b)^2 + (c-c)^2}$$
$$= \sqrt{a^2 + 0^2}$$
$$= \sqrt{a^2}$$
$$= a$$

$$ZW = \sqrt{(b-0)^2 + (c-0)^2}$$
$$= \sqrt{b^2 + c^2}$$

Diagonal lengths:

$$XZ = \sqrt{(a-b)^2 + (c-0)^2}$$
$$= \sqrt{a^2 - 2ab + b^2 + c^2}$$

$$WY = \sqrt{(a+b-0)^2 + (c-o)^2}$$
$$= \sqrt{(a+b)^2 + (c)^2}$$
$$= \sqrt{a^2 + 2ab + b^2 + c^2}$$

Find the sum of the squares of the sides:

$$WX^2 + XY^2 + YZ^2 + ZW^2$$
$$= (a)^2 + \left(\sqrt{b^2 + c^2}\right)^2 + (a)^2 + \left(\sqrt{b^2 + c^2}\right)^2$$
$$= a^2 + b^2 + c^2 + a^2 + b^2 + c^2$$
$$= 2a^2 + 2b^2 + 2c^2$$

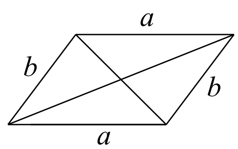
Find the sum of the squares of the diagonals:

$$WY^2 + XZ^2$$
$$= \left(\sqrt{a^2 + 2ab + b^2 + c^2}\right)^2 + \left(\sqrt{a^2 - 2ab + b^2 + c^2}\right)^2$$
$$= a^2 + 2ab + b^2 + c^2 + a^2 - 2ab + b^2 + c^2$$
$$= 2a^2 + 2b^2 + 2c^2$$

$\therefore WX^2 + XY^2 + YZ^2 + ZW^2 = WY^2 + XZ^2$

This is known as the Parallelogram Law.

Parallelogram Law: The sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.



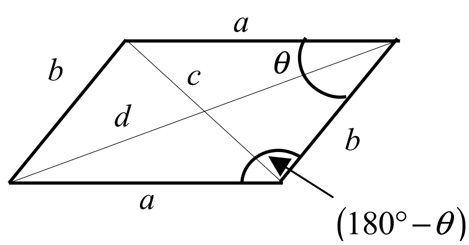
If the parallelogram shown on the left has diagonals of lengths c and d, then $2a^2 + 2b^2 = c^2 + d^2$

Note: If the parallelogram is a rectangle, then diagonals are congruent and $c = d$. If we substitute c in place of d in the equation above, we see that

$$2a^2 + 2b^2 = c^2 + d^2$$
$$2a^2 + 2b^2 = c^2 + c^2$$
$$2a^2 + 2b^2 = 2c^2$$
$$a^2 + b^2 = c^2$$

which is the Pythagorean Theorem.

B. Proof using Law of Cosines



Given the parallelogram on the left, let c and d be the lengths of the diagonals.

Prove: $2a^2 + 2b^2 = c^2 + d^2$

We can use the Law of Cosines to find expressions for the squares of lengths of the diagonals:

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$
$$d^2 = a^2 + b^2 - 2ab\cos(180° - \theta)$$

The sum of the squares of the diagonals is

$$c^2 + d^2$$
$$= a^2 + b^2 - 2ab\cos(\theta) + a^2 + b^2 - 2ab\cos(180° - \theta)$$
$$= 2a^2 + 2b^2 - 2ab\cos(\theta) - 2ab\cos(180° - \theta)$$
$$= 2a^2 + 2b^2 - 2ab\cos(\theta) - 2ab[\cos 180° \cdot \cos\theta + \sin 180° \cdot \sin\theta]$$

The angle sum and difference identity for cosine is: $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

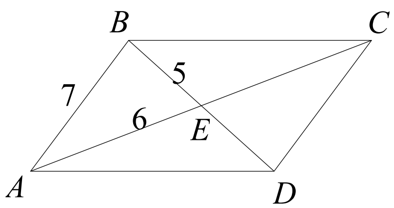
$$= 2a^2 + 2b^2 - 2ab\cos(\theta) - 2ab[(-1)\cdot\cos\theta + (0)\cdot\sin\theta]$$
$$= 2a^2 + 2b^2 - 2ab\cos(\theta) - 2ab(-\cos\theta)$$
$$= 2a^2 + 2b^2 - 2ab\cos(\theta) + 2ab\cos(\theta)$$
$$= 2a^2 + 2b^2$$

$\therefore\ 2a^2 + 2b^2 = c^2 + d^2$

Warm-Up

CST 7.0

If $ABCD$ is a parallelogram, what is the length of $\overline{BD}$?



A. 10

B. 11

C. 12

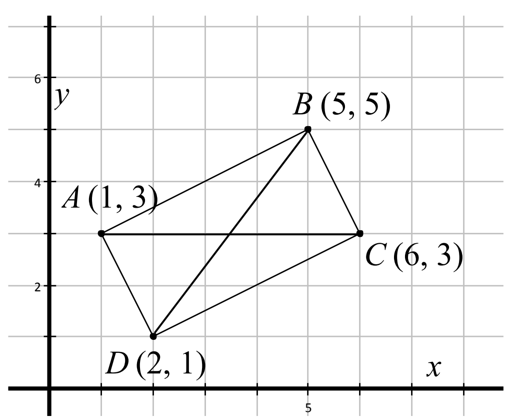
D. 14

Other

Is it possible to find the length of $\overline{BC}$ in the parallelogram shown in problem 1? If so, how?

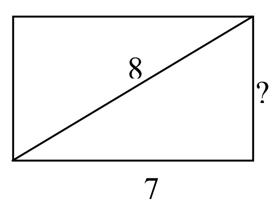
Current 7.0

Find the lengths of the sides and diagonals of the quadrilateral below. What kind of quadrilateral is *ABCD*?
How do the lengths you found help you determine this?



Review 15.0, Gr. 7 MG 3.3

What is the height of this rectangle?



A. 1 unit

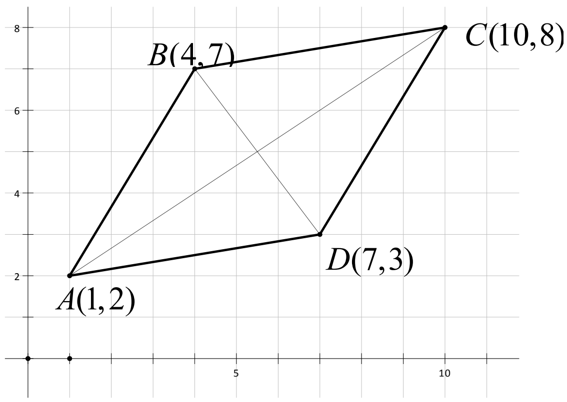
B. 6 units

C. $\sqrt{15}$ units

D. $\sqrt{113}$ units

The Parallelogram Law

1) Find the lengths of the sides and diagonals for the parallelogram below.



2) Find the sum of the squares of the sides.

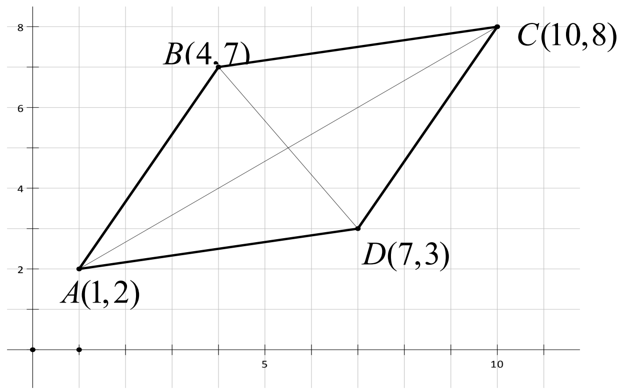
3) Find the sum of the squares of the diagonals.

4) How do the answers to 3 and 4 compare?

Conjecture: The sum of the squares of the sides of a parallelogram _________________ the sum of the squares of the diagonals.

Introductory Handout Key:

1) Find the lengths of the sides and diagonals for the parallelogram below.



Sides:

$$AB=\sqrt{(4-1)^2+(7-2)^2} =\sqrt{9+25} =\sqrt{34}$$

$$BC=\sqrt{(10-4)^2+(8-7)^2} =\sqrt{36+1} =\sqrt{37}$$

$$CD=\sqrt{(10-7)^2+(8-3)^2} =\sqrt{9+25} =\sqrt{34}$$

$$DA=\sqrt{(7-1)^2+(3-2)^2} =\sqrt{36+1} =\sqrt{37}$$

Diagonals:

$$AC=\sqrt{(10-1)^2+(8-2)^2} =\sqrt{81+36} =\sqrt{117}$$

$$BD=\sqrt{(4-7)^2+(7-3)^2} =\sqrt{9+16} =\sqrt{25} =5$$

2) Find the sum of the squares of the sides.

$$AB^2+BC^2+CD^2+DA^2$$
$$=\left(\sqrt{34}\right)^2+\left(\sqrt{37}\right)^2+\left(\sqrt{34}\right)^2+\left(\sqrt{37}\right)^2$$
$$=34+37+34+37$$
$$=142$$

3) Find the sum of the squares of the diagonals.

$$AC^2+BD^2$$
$$=\left(\sqrt{117}\right)^2+5^2$$
$$=117+25$$
$$=142$$

4) How do the answers to 3 and 4 compare?

The answers are the same.

Conjecture: The sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.

Student Exploration Handouts

Demonstrating the Parallelogram Law

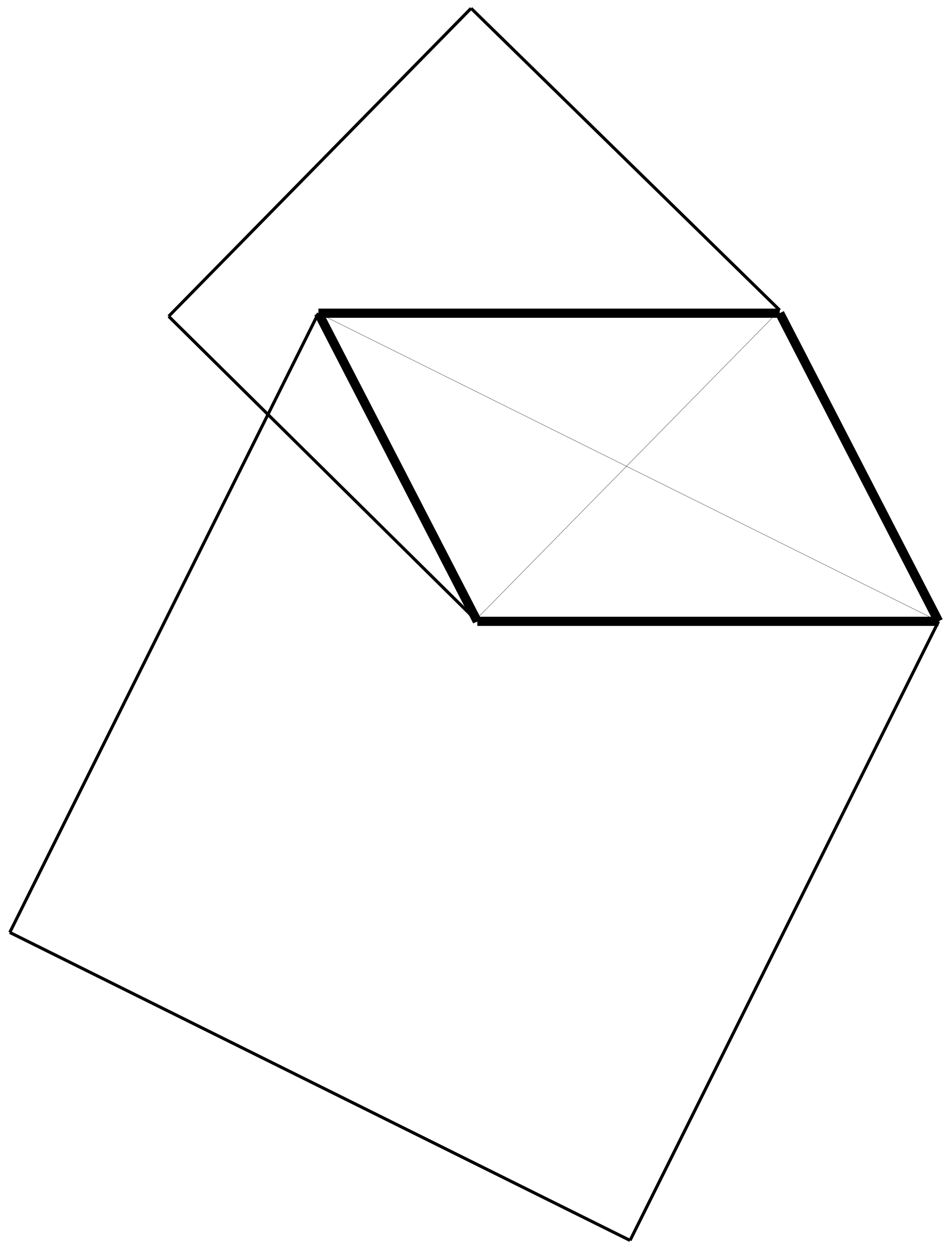
Materials: Scissors, glue stick, two different colored highlighters

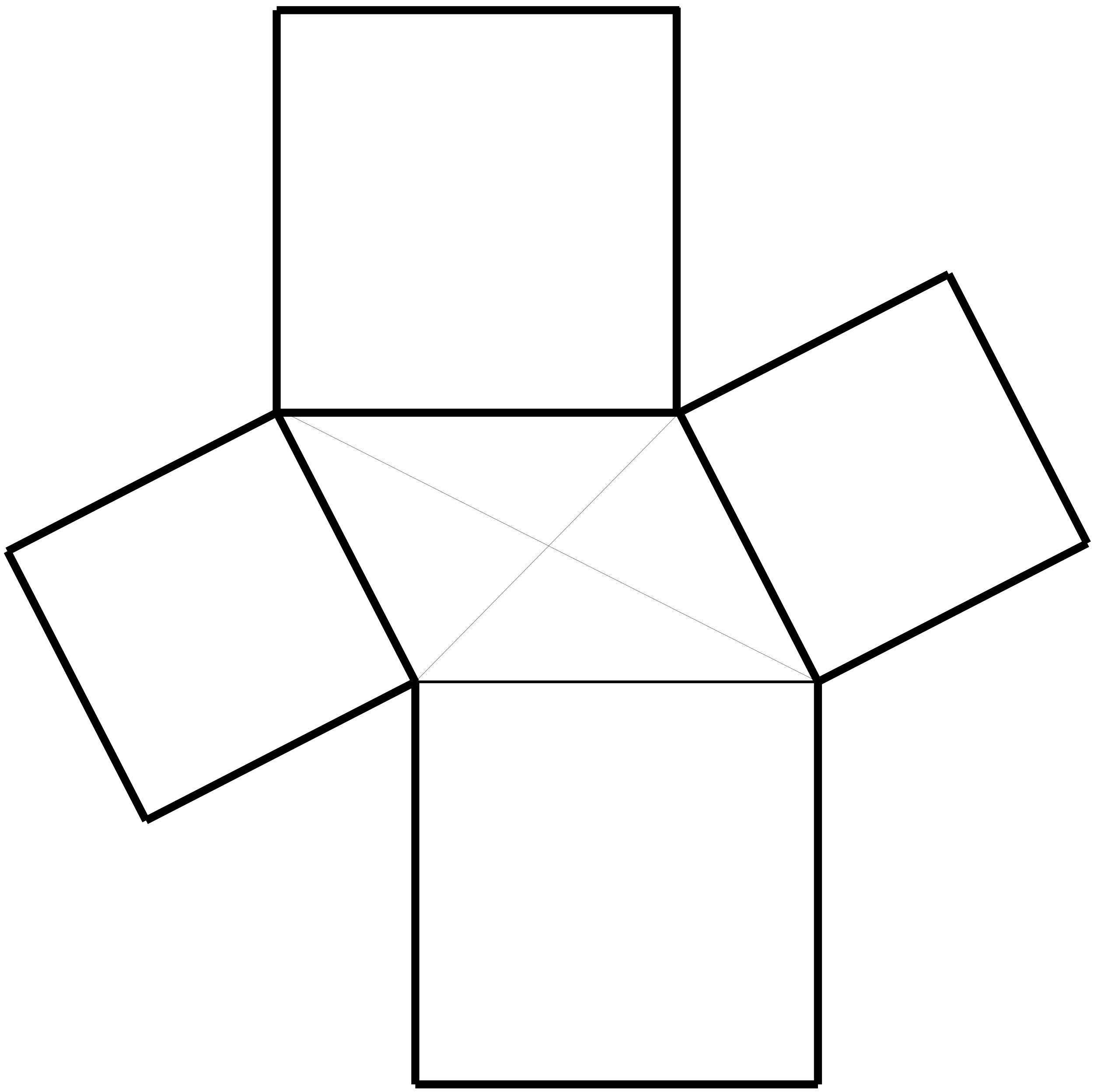
You will demonstrate the Parallelogram Law by cutting up the squares formed along the diagonals of a parallelogram and fitting them into the four squares around the sides of the parallelogram.

- On your handouts, label the longer sides of the parallelogram *a* and the shorter sides *b*. Label the shorter diagonal *c* and the longer diagonal *d*.
- On the Cut-Outs page, color both of the squares labeled "Diagonal *c*" one color. Color both of the squares labeled "Diagonal *d*" a second color.
- Cut out one each of the "Diagonal *c*" and "Diagonal *d*" squares. Glue them next to the diagonals on Page 1 of your handout.
- Cut out the other "Diagonal *c*" and "Diagonal *d*" squares. Cut them apart in any way you want and glue the pieces into the squares along the sides of the parallelogram on page 2. (Note: You might have to cut some very small pieces, so it will be hard to make it exact.)

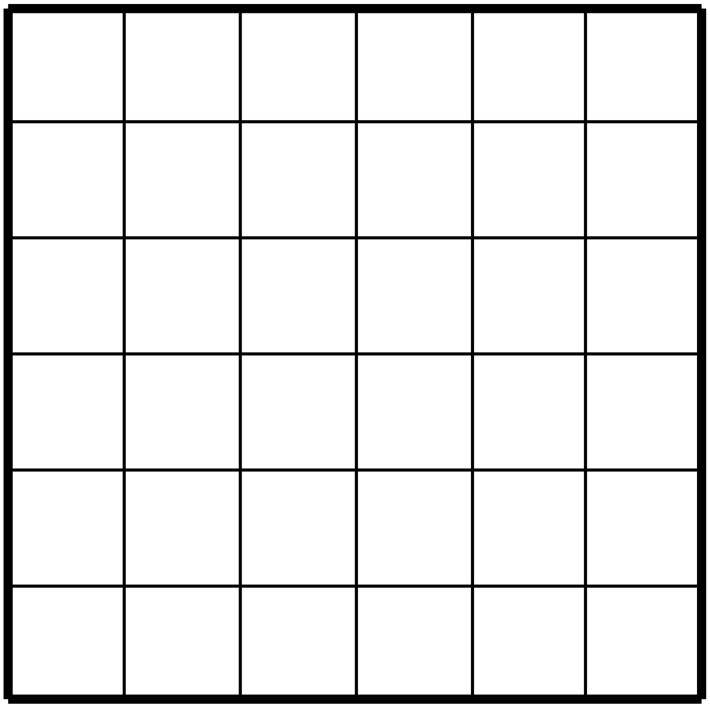
Does it appear to be true that the squares of the diagonals will fit into the squares along all four sides?

Parallelogram Law: The sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.

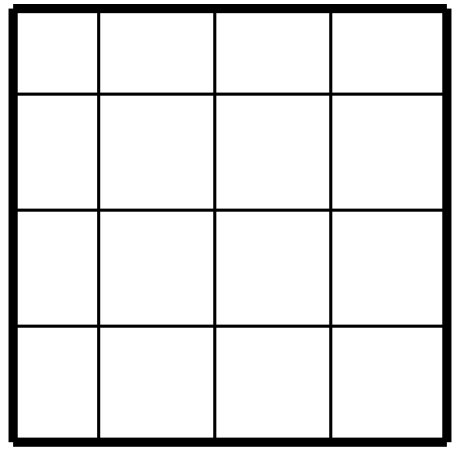




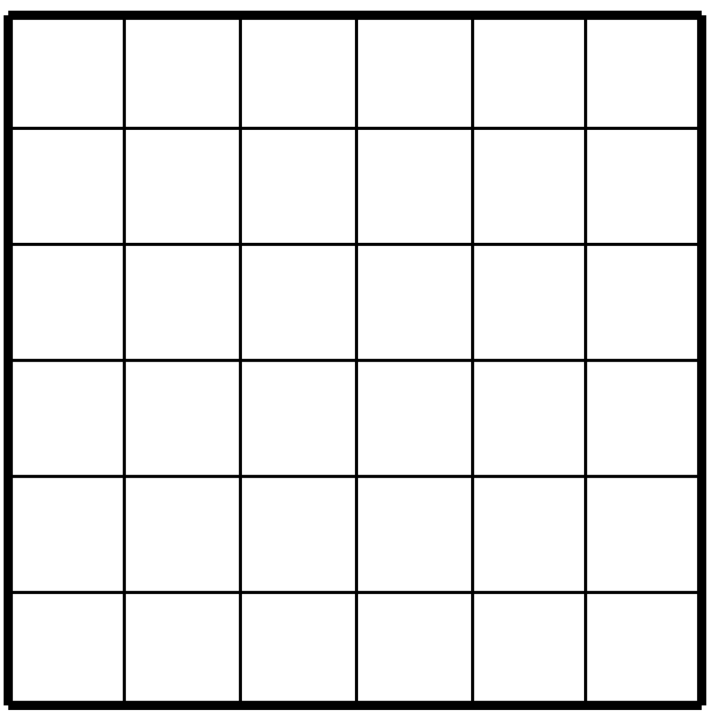
The Parallelogram Law - Cut Outs



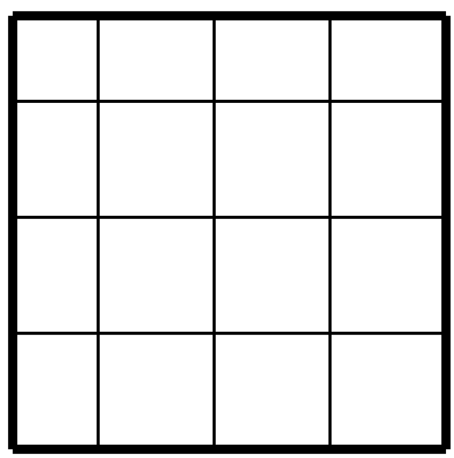
Diagonal *d*



Diagonal *c*



Diagonal *d*



Diagonal *c*